Convergence Acceleration of Relaxation Solutions for Transonic Flow Computations

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The problem of how to speed up the convergence of currently available iterative methods for transonic flow computations with minimal alterations in computer programing and storage is considered. A cyclic iterative procedure applying nonlinear sequence transformations akin to those of Aitken and Shanks is developed. Based on the "power method," the errors in these sequence transformations are studied. Examples testing the procedure for model Dirichlet problems and for transonic thin airfoil problems show that savings in computer time of a factor of two to five, or more, is generally possible, depending on accuracy requirements and the particular iterative procedure used.

I. Introduction

MANY current methods of fluid dynamic computations make use of relaxation procedures. There are several aspects of the computation that considerably limit the usefulness and potentiality of these programs. One is the slow convergence with respect to iterations of the flowfield calculation, and, hence, costly computer time. This paper presents studies on how to speed up the convergence of currently available iterative procedures with minimal alterations in computer programing and storage requirements.

To see the need of the convergence acceleration, we may take as an example the finite-difference solution to Dirichlet's problem on a unit square. The convergence rate of practical iterative procedures (Jacobi, Gauss-Seidel, Successive Over-Relaxation, or Block Symmetric Successive Over-Relaxation)[1-4] depends on the magnitude of the eigenvalue of the largest modulus of the corresponding iterative matrix, denoted here by $|\lambda_1|$, i.e., spectral radius. The error of the solution at the kth iterations is, in most cases, gaged by $|\lambda_1|^k$ (cf. Refs. 2 and 5). The need for improvement is apparent from the fact that $|\lambda_1|$ tends to unity as the mesh size vanishes. Thus, for an accurate solution, the convergence is very slow.

For certain well-ordered sparse matrices with constant coefficients, the use of direct inversion methods can be quite efficient (see Ref. 6); with suitable alterations of the difference equations, a direct elliptic solver has been effectively applied in combination with relaxations to the transonic small-disturbance equation by Martin and Lomax.[7-9] Instead of altering the basic iterative procedure in the existing programs, we will adopt a cyclic acceleration procedure in which weakly nonlinear sequence transformations closely related to those of Shanks[10] and Aitken[11,12] are applied at the conclusion of each cycle of iterations, generating initial data for the next cycle.

The storage and the number of arithmetic operations required by current computer programs[13-21] for transonic flow computations are low enough to make the computations possible even with modest computers. The computer time for 400-1000 iterations required in the more complicated problems may still demand 0.5 to 2 hr on an IBM 370/158 or 360/44, and 10-40 min on a CDC-6600. Thus the use of acceleration techniques is certainly worthwhile, especially if one has a great many problems to solve. Note, however, that the iterative matrix for the quasilinear, mixed-flow problem does not lend itself to an *a priori* determination of its eigenvalues. The proper choice of the relaxation parameter ω for the relaxation method in this case has been mostly guess work.

One practical method of reducing the number of iterations for the relaxation problem is grid refinement, used by South and Jameson in transonic flow calculation.[15] The idea is to converge a calculation on a coarse mesh and use the result as an initial guess for a refined mesh. However, the fundamental problem of slow convergence eventually appears as the mesh is refined. This difficulty may be overcome in some cases with the multilevel grid method,[14] where the coarse grids are used to filter out residuals with longer wavelengths corresponding to the lower eigenvalues.

In the cited work of Martin and Lomax,[7,8] an algorithm using Padé fractions has also been employed to accelerate their iterative method; in effect, Aitken's δ^2 process was applied independently to each grid point with an implementing algorithm that stipulates a nearly geometric sequence. In our study, the pointwise application of the δ^2 process is found to be less effective (cf. remarks in Secs. III E and IV).‡

The kinship of our acceleration technique with the Shanks nonlinear transformation[10] and Aitken's δ^2 process is noteworthy,[11,12] but there are essential differences. These are clarified in the next section, where the basis of the power method is introduced (Sec. II B) and the importance for allowing close spacings between successive eigenvalue moduli is amplified (Sec. II C). The main theoretical content of our method is presented in Sec. III where the errors in the power method are analyzed and the sequence transforms underlying our cyclic iterative method are derived. Numerical experiments with the cyclic acceleration are applied to transonic flow problems in Sec. V.

II. Remarks on Sequence Transformations and Power Method

A. Transformation of Sequence

The use of transformations to improve convergence characteristics of sequences is not unfamiliar in fluid

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‡For the more recent development in Padé approximants and related methods, see Ref. 40 where the present work as well as that of Martin and Lomax are also reviewed.

mechanics and applied mathematics.[22-24] One class of these that bears a kinship to the key equations of our method is the nonlinear transformations of Shanks,[10] and the related Padé rational fractions.[25] From a sequence, say ϕ_1, ϕ_2,..., ϕ_{k-1}, ϕ_2, ϕ_{k+1},..., the transformation gives a new sequence closer to the limit ϕ. The simplest among these is the e_1 transform of Shanks

$$\phi'_{k+1} = \frac{\phi_{k-1}\phi_{k+1} - \phi_k^2}{\phi_{k-1} - 2\phi_k + \phi_{k+1}} \tag{1}$$

which predicts the limit ϕ from three successive iterates ϕ_{k-1}, ϕ_k, and ϕ_{k+1}. Equation (1) has been found and applied independently in many earlier works, and the algorithm is often referred to as Aitken's δ^2 process (see Refs. 4, 5, 10-12); it could also be viewed as a derivative-free variant of Newton's method or the method of false position (see for example, Ref. 5, pp. 96-109).§

As a predictor of the limit ϕ for an iterative solution to a nonlinear *scalar* equation $\phi_{k+1} = g(\phi_k)$, Eq. (1) is subject to an error comparable to the squre of $\epsilon_k \equiv \phi_k - \phi$ under a nonvanishing $g'(\phi)$ and is, in fact, exact if the equation is linear. However, for the iterative *matrix* equations of interest here, these estimates are not strictly correct even in the linear case. The validity (and accuracy) of Eq. (1) and similar transforms must be established on an entirely different basis. Formally the first-order transform in the present method could be regarded as a variant of those of Aitken and Shanks; however, a point-by-point application of Eq. (1) in the manner of Aitken to the iterative line-relaxation solution of a matrix equation proves to be unreliable and uneconomical. (See Secs. III and V below; neither Aitken[12] nor Wilkinson[4] have found their transforms very successful with matrix solutions.) We note in passing that Wilkinson,[4] following Aitken,[12] applied Eq. (1) to approximate the lowest-order eigenvector of a matrix, and claimed an $\|\epsilon_k\|^2$ accuracy; but the proof (which overlooks the contribution from the third eigenvectors, Ref. 4, p. 578) is itself in error.

Considering the sequence $\{\phi_k\}$ as the k-term partial sum of a series of an analytic function, Shanks identifies one of his transformed sequence $\{e_n(\phi_k)\}$ with the nth row in the upper triangle of the Padé Table.[10] (For Padé fractions and further relations to Shanks' transforms, see Refs. 26-28). An important observation motivating the work of Shanks is that the transformed sequence $\{e_n(\phi_k)\}$ represents exactly the limit of a sequence $\{\phi_k\}$, if ϕ_k has (precisely) the *transient* behavior for successive k

$$\phi_k = \phi + \sum_{i=1}^{n} a_i q_i^k \tag{2}$$

where a_i and q_i are constants. It is apparent that convergence requires $|q_i| < 1$¶ and that, for a sequence from the partial sum of a geometric series ($a_i = 0$, $i \neq 1$), the e_1 transform of Shanks yields the exact limit. The *stipulated* exponential transient, Eq. (2), is not a general one, for there is no apparent reason that the iterates of a general scalar equation cannot approach its limit algebraically instead. Interestingly, for the iterative solution to a matrix equation, a transient similar to Eq. (2) *does* apply to each *component* of the solution near the convergence limit (cf. Secs. II and III) although this point may not have been recognized by Shanks.

§Aitken also studied successive transformations of new sequences, i.e., e_1^m, cf. Refs. 11 and 12.

¶We have not explored here the potentiality of Shanks' transformation to convert a divergent sequence to a convergent one, corresponding to Shanks' notion of an "antilimit." This fact may be quite useful; for example, if $|\lambda_1| > 1$ but $|\lambda_i| < 1$, $i = 2, 3, \ldots, N$, Eq. (17) in Sec. III A still holds.

B. Linearized System and Power Method

In the relaxation solution to the difference equations of interest, the unknown ϕ and its kth iterate ϕ_k are vectors with components equal in number of the total number of grid points N. The iterative matrix equation of interest is

$$\phi_{k+1} = g(\phi_k) \tag{3}$$

where the function g depends on the difference equations and the iterative procedure used. In approaching the convergence limit, the error vector

$$\epsilon_k = \phi_k - \phi \tag{4}$$

satisfies a linearized matrix equation

$$\epsilon_{k+1} = Q\epsilon_k \tag{5}$$

where Q is the Jacobian matrix of g, with a remainder comparable to the *square* of (some norm of) ϵ_k, assuming that g is well behaved and independent of k. It may suffice, therefore, to analyze the error vector on the basis of this linear recursion relation, with a second-order accuracy. The equations governing the limit solution ϕ may, however, be nonlinear.

We note in passing that Eq. (5) is equivalent to a discrete version of a time-dependent system, say,

$$C\dot{\phi} = A\phi \tag{6a}$$

for arbitrary matrices C and A, and a time step Δt, so long as

$$Q = \exp\{\Delta t\, C^{-1}A\} \tag{6b}$$

Returning to the linear iterative system, Eq. (5), the matrix Q has generally a set of eigenvalues λ_i with corresponding eigenvectors v_i

$$Q v_i = \lambda_i v_i \qquad i = 1,2,3,4,\ldots \tag{7}$$

We assume for the moment that the λ_i's are distinct and can be ordered according to their moduli as $|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_{N-1}| > |\lambda_N|$. The error vector of the initial data may then be represented by

$$\epsilon_0 = \phi_0 - \phi = \sum_{i=1}^{N} \alpha_i v_i \tag{8}$$

Repeated iterations on Eq. (8) with Eq. (5) yields the error vector at the kth iteration

$$\epsilon_k = \phi_k - \phi = \sum_{i=1}^{N} \alpha_i v_i \lambda_i^k \tag{9}$$

This is the main base for the power method of Fadeev and Fadeeva.[29] Obviously, convergence requires $|\lambda_1| < 1$ and, near the limit, the error vector is dominated by the lowest eigenvector v_1 as

$$\epsilon_k = \alpha_1 v_1 \lambda_1^k + O(\alpha_2 v_2 \lambda_2^k) \tag{10}$$

Omitting the remainder proportional to λ_2^k, Eq. (10) can be applied to three successive iterates and recovers readily the Aitken-Shanks transformation, Eq. (1). This should not be surprising because, for $n < N$ and writing q_i as λ_i, the "transient" of Shanks, Eq. (2), for which the e_n transform gives the exact limit is identifiable with the first n terms of the error vector in Eq. (9). In other words, the e_n transform applied to matrix solutions finds a theoretical basis in the power method, with an error expected to be proportional to

$$(\lambda_{n+1})^k \ll 1 \tag{11}$$

Although the approach to the limit is exponential in k, according to Eq. (9), $|\lambda_1|$, $|\lambda_2|$, etc. are very close to unity in most problems of interest. This makes the convergence extremely slow; it also makes the error estimates for the e_n transform, Eqs. (10) and (11), unreliable (see Sec. III below). To illustrate this behavior of λ_i's and their dependence on the mesh size, we examine a model elliptic problem.

C. Eigenvalues of Iterative Matrix: Model Problem

Consider the finite-difference relaxation solution to the Laplace equation in a rectangular domain with ϕ prescribed on the boundary (a two-dimensional Dirichlet problem). Using a uniform mesh size, the length of two sides of the rectangle are taken to be I and J units, respectively, with $I>J$. The iterative matrix equation of $N=(I-1)\cdot(J-1)$ unknowns resulting from the central-difference scheme, using standard iterative techniques, can be written as

$$\phi_{k+1}=Q\,\phi_k+d \tag{12}$$

consistent with Eq. (5). Following Young,[2] the eigenvalue is denoted by μ_{pq} for the Jacobi (J) method, and by λ_{pq} for the Gauss-Seidel (GS) method. In terms of the mesh size $\Delta x=\Delta y=h$, the two largest eigenvalues of the J method may be represented for small h as (Ref. 2, pp. 71-73 and 131)

$$\mu_1\sim 1-\frac{1}{2}\pi^2h^2 \qquad \mu_2\sim 1-\frac{5}{4}\pi^2h^2 \tag{13a}$$

and, for the GS method, as

$$\lambda_1\sim 1-\pi^2h^2 \qquad \lambda_2\sim 1-\frac{5}{2}\pi^2h^2 \tag{13b}$$

If $|\lambda_1|^k=e^{k\ln|\lambda_1|}$ is taken as an estimate of norm $\|\epsilon_k\|$, the GS method would then take $1/\pi^2h^2$ or N/π^2 iterations to reduce $\|\epsilon_k\|$ by a factor of e^{-1}; the corresponding iterations for the J method is $2N/\pi^2$. If the Shanks-Aitken transform can be used, the error norm becomes $|\lambda_2|^k=e^{k\ln|\lambda_2|}$, according to Eq. (11); comparing λ_2 *with* λ_1 and μ_2 with μ_1 in Eq. (13), the convergence rate is seen to increase 2.5 times in either case. The line-relaxation version of the GS or J method converge twice as fast.[2,3]

Over-relaxation applied to GS methods or its line relaxation version, using $\phi_{k+1}=\phi_k+\omega(\phi'_{k+1}-\phi_k)$, increases the convergence rate in the model Dirichlet problem, provided $1>\omega>2$. For this and other more general matrix equations, Young shows that an optimum ω exists between 1 and 2 for the SOR method, at which the spectral radius reaches its smallest value satisfying (cf. Ref. 2, pp. 172, 173)

$$1-|\lambda_1|=O(\pi h) \tag{14}$$

This would lead to an order-of-magnitude saving in iterations; but this possibility is not stipulated here, because the optimum ω cannot be inferred for the nonlinear problem (for ϕ) of interest, and also because the neighborhood of the optimum ω for which Eq. (14) holds is very narrow.

Young also observes that for $0<\omega<\omega_{\text{opt}}$ the spectral radius decreases monotonically with increasing ω toward ω_{opt} (where there is a square root singularity) and that the spectral radius increases linearly as $(\omega-1)$ for $\omega_{\text{opt}}<\omega<2$. Except when ω is very close to 2, say $\omega=2-0(h^2)$, these observations and Eq. (14) indicate that the problem of slow convergence is more serious in the range $0<\omega<\omega_{\text{opt}}$.

An important feature of the SOR solution to the model Dirichlet problem is that, for $\omega_{\text{opt}}<\omega<2$, *all* eigenvalues have the same modulus, i.e., $|\lambda_1|=|\lambda_2|=\ldots=|\lambda_N|$ (cf. Ref. 2, pp. 203-206.) This shows clearly that the applicability of acceleration techniques based on the power method Eqs. (9-11) is limited only to

$$\omega<\omega_{\text{opt}}. \tag{15}$$

for which the first few eigenvalues are close to unity and satisfy

$$1-\lambda_i=O(h^2) \tag{16}$$

III. Errors in the Power Method and Nonlinear Transformations

For large-scale computations involving very fine grids ($h\ll 1$), the accuracy and convergence rate of the Shanks e_n transform cannot be regarded as well-founded. This is because the very fact that

$$\lambda_1-1=O(h^2)\neq 0 \qquad \lambda_i-\lambda_1=O(h^2) \tag{17}$$

where $i=2,3,\ldots,n\ll N$, brought out in Sec. IIC for the model elliptic problem, has not been allowed for in classical works.[1,3-5,29] This unaccounted fact could change the error estimate for the transforms [cf. Eq. (19) below]. More specifically, a factor $(1-\lambda)^{\sigma}$ with an undetermined exponent σ may enter in the error estimate, which would be compounded in successive applications. In the following, we shall establish the transforms on a firmer basis, by taking into account Eq. (17), including the case with repeated λ_i's.

A. First-Order Transform

We shall first consider cases in which the eigenvalues are distinct. For the special case with a single dominant eigenvalue, i.e., $|\lambda_1|>|\lambda_2|>|\lambda_3|$ etc., the formal basis for the δ^2 process derived from the power method is usually taken as $e_{k+1}\simeq\lambda_1 e_k$. The error of this equation may be studied from the (exact) relation based on Eq. (10)

$$e_{k+1}-\lambda_1 e_k=\sum_{i=2}^{N}\alpha_i v_i\lambda_i^k(\lambda_i-\lambda_1) \tag{18}$$

From this, one may formally predict the limit ϕ from two iterates and λ_1, using the equation

$$\phi=\phi_k+\frac{\phi_{k+1}-\phi_k}{1-\lambda_1}-\sum_{i=2}^{N}\alpha_i v_i\lambda_1^k\left(\frac{\lambda_i-\lambda_1}{1-\lambda_1}\right) \tag{19}$$

where the last term gives a remainder at most of the order λ_2^k. The need for a critical analysis is seen from the appearance of $(1-\lambda_1)$ in the denominators. Larger errors may thus arise, depending on the method for estimating λ_1.

The simplest way of inferring λ_1 is to determine it from *three successive* iterates via Eq. (18) for a chosen (reference) component of ϕ_k's, say ϕ_k^*, at sufficiently large k,

$$\lambda_1=\lambda_1^*+\Delta\lambda_1 \tag{20a}$$

with

$$\lambda_1^*=(\phi_{k+1}^*-\phi_k^*)/(\phi_k^*-\phi_{k-1}^*) \tag{20b}$$

$$\Delta\lambda_1=(\phi_k^*-\phi_{k-1}^*)\sum_{i=2}^{N}\alpha_i v_i\lambda_i^{*\,k-1}(\lambda_i-\lambda_1)(\lambda_i-1)$$

$$\sim\frac{\alpha_2 v_2}{\alpha_1 v_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k-1}\frac{(1-\lambda_2)(\lambda_2-\lambda_1)}{(1-\lambda_1)} \tag{20c}$$

By virtue of the extra factor $(\lambda_2-\lambda_1)=O(h^2)$ in Eq. (20c), λ_1^* as an estimate for λ_1 should be reasonably accurate. An equivalent estimate, which will have less problems than λ_1^* with rounding error and with the sensitivity to the component chosen for reference, is

$$\bar{\lambda}_1=\Sigma|\phi_{k+1}-\phi_k|/\Sigma|\phi_k-\phi_{k-1}| \tag{21}$$

where Σ signifies the summation over all components. Other implementations on the λ_1 estimates are discussed in Sec. IID.

With λ_1 from Eq. (20), the limit ϕ may now be predicted as

$$\phi=\phi_k+\frac{\phi_{k+1}-\phi_k}{1-\lambda_1^*}+\Delta_k\phi \tag{22a}$$

with

$$\Delta_k\phi\simeq\sum_{i=2}^{N}\lambda_i^{k-1}\left(\frac{\lambda_i-\lambda_1}{1-\lambda_1}\right)\left[\frac{\alpha_i v_i^*\lambda_1(\lambda_i-1)}{1-\lambda_1}\cdot\frac{\phi_{k+1}-\phi_k}{\phi_{k+1}^*-\phi_k^*}+\alpha_i v_i\lambda_i\right]=O(\alpha_2\lambda_2^k) \tag{22b}$$

where Eq. (17) has been used. A similar estimate of $\Delta_k\phi$ can be obtained if λ_1^* in Eq. (22) is replaced by $\bar{\lambda}_1$. The transform based on Eq. (22a) with λ_1^* defined by Eq. (20a) or its equivalent, will be referred to as the *first-order* transform.

B. Second- and Higher-Order Transforms

Since complex eigenvalues of a real matrix occur in pairs, one must allow for cases in which two or more eigenvalues are equally dominant, e.g., $|\lambda_1|=|\lambda_2|$, even if they are distinct. The need also arises if $|\lambda_2|$ is too close to $|\lambda_1|$ for the first-order transform to be useful.

In Ref. 30,** we obtained a transform to replace Eq. (22), making use of the first n eigenvectors to filter out the error. We assumed that all λ_i's are distinct, although some of their moduli may be equal. The limit ϕ is predicted from n successive iterates of the same component as (with $p_n=1$)

$$\phi=\phi_k+\frac{\sum_{j=0}^{n}p_j(\phi_{k+j}-\phi_k)}{\sum_{j=0}^{n}p_j}+\Delta_k^n\phi \tag{23}$$

where p_j are explicit functions of the λ_i's and can be used in place of λ_i's; the remainder is of the order $\alpha_{n+1}\lambda_{n+1}^k$.

C. Case with Repeated Roots

If two or more eigenvalues coincide, and if Q is not diagonalizable, Eq. (9) of the power method is not applicable because of the lack of a complete set of independent eigenvectors. On p. 6 of Ref. 30, we proved that the nth-order transform holds for repeated roots, although the derivation in this case is different from the case of distinct eigenvalues. The order of the remainder is modified, but Eq. (23) remains unchanged.

D. Implementation

Instead of applying the first-order transform to three successive iterates, one may apply it to ϕ_{k-m}, ϕ_k, and ϕ_{k+m}. In this case, Eq. (22) is replaced by

$$\phi=\phi_k+\frac{\phi_{k+m}-\phi_k}{1-\lambda_1^m} \tag{24a}$$

with

$$(\lambda_1^m)\simeq(\lambda_1^m)^*=(\phi_{k+m}^*-\phi_k^*)/(\phi_k^*-\phi_{k-m}^*) \tag{24b}$$

where the remainders in ϕ and λ_1^m are the same as in Eqs. (20b) and (22b) with λ_1 and λ_j replaced by λ_1^m and λ_j^m, except the λ_j in λ_j^{k-1}. The *first*-order transform with an *even* m is applicable to the case with $\lambda_1=-\lambda_2$ (even though $|\lambda_1|=|\lambda_2|$), the error of the transform belongs, in this case, to the order λ_3^k. Similarly, the nth-order transform may also be applied to iterates separated by m. Generally, the *order* of the error in ϕ remains at λ_{n+1}^k under condition Eq. (17), independent of m.

A key to the successful application of the first-order transform is the provision of an accurate and reliable estimate for the eigenvalue λ_1. One advantage of using Eq. (24) with $m>1$ is, in fact, the reduction in the sensitivity of the first-order transform, Eq. (22), with respect to the error in estimating λ_1. We observe in this regard that not only is any error in λ_1 amplified by the factor $(1-\lambda_1)^{-2}=O(h^{-4})$ in the transform, but the computed value of λ_1^* varies considerably from component to component during the transient. In addition, there is a serious problem with rounding error in the computation of λ_1^* since, near the convergence limit, both the numerator and denominator of λ_1^* may not remain large compared with rounding errors in certain computers (IBM 360/44, 370/158, etc.) using single-precision arithmetic. In this regard, the alternative of estimating λ_1 by $\bar{\lambda}_1$ from Eq. (21), in which both the numerator and denominator are large and represents an average among the N components, gives a better convergence behavior. Another method for estimating λ_1 which provides even better results is to compute λ_1 as a quotient of two inner products

$$\bar{\bar{\lambda}}_1^m=\delta_k^T\delta_{k+m}/\delta_k^T\delta_k \tag{25}$$

where δ_k is the N-component vector $\delta_k=\phi_k-\phi_{k-m}$. This estimate for λ_1, whose accuracy is comparable to $\bar{\lambda}_1$ is comparable to the "Rayleigh quotient" $\delta_k^TQ\delta_k/\delta_k^T\delta_k$ but requires only $O(N)$ multiplications (as compared to $O(N^2)$ multiplication otherwise).

Similar comments apply to the problems of estimating p_j's or λ_j's for the higher-order transform.†† Estimates of p_0 and p_1 for the second-order transform similar to $\bar{\lambda}_1$ can be obtained, for example, from $p_0\delta_k^T\delta_k+p_1\delta_k^T\delta_{k+1}+\delta_k^T\delta_{k+2}=O$ and similar equations.

E. Cyclic Acceleration Method

The first- and higher-order transforms, Eqs. (22a) and (23), can be used to improve the accuracy of the relaxation solution at the conclusion of a large number of iterations, as in Lyusternik's work,[32,4] or to convert $\{\phi_k\}$ to a new sequence closer to the limit.

In the present work, these transforms are used as a part of an iterative algorithm: the procedure consists of several cycles, each of which makes k' (10 to 30) iterations on the (nonlinear) algebraic system; the transformation is applied at the end of each cycle to yield an estimate of the limit, which is used as initial data for the next cycle. The error norm is reduced by a factor of $\lambda_{n+1}^{k'}$ at the end of each iterative cycle (which is carried into the coefficient α_{n+1} in the next approximation), the error after σ cycles is $O(\lambda_{n+1}^k)$, where k is the total number of iterations $\sigma k'$. The convergence rate is accordingly unaffected by the subdivision into cycles.

Figure 1 illustrates the cyclic application of the first-order transform for a single component. The application of the transform at the end of the cycle, say the $k+m$ iterate, requires the value of ϕ_{k+m}, and the *stored* value of the same component from a previous iteration, ϕ_k. If λ_1^m is to be determined as $(\lambda_1^m)^*$, Eq. (24b), only the value of the reference component ϕ_{k-m}^* is needed and carried as a *single* datum (for the entire matrix). If λ_1^m is chosen to be $\bar{\bar{\lambda}}_1^m$, one can compute $\Sigma\,|\delta_k|$ and store it as a single datum along with ϕ_k; $\bar{\bar{\lambda}}_1^m$ is computed after the whole field of ϕ_{k+m} has been obtained. The use of the $\bar{\bar{\lambda}}_1^m$, Eq. (25), requires storing the whole field of δ_k along with that of ϕ_k. When the second-order transform is used (for predicting ϕ after the $k+2m$

**The part "$i=1$" under the summation sign in Eq. (3.8b) of Ref. 30 should be replaced by "$i=1+n$."

††In general, one has to solve a system of n linear algebraic equations for the p_j's.

iteration), full storage for the two vectors ϕ_k and ϕ_{k+m} is needed; additional storage for two vectors δ_{k-m} and δ_k is required if the inner-product quotient forms for p_0 and p_1 are used. The use of the latter quotient often yields a smoother approach to the limit and seems to be worthwhile.

The additional arithmetical operations resulting from the application of the acceleration technique is negligibly small. For the most complicated version of our second-order transform, the computation time for a specified number of iterations seldom exceeds 5% of the amount for the corresponding unaccelerated case.

We observe in passing that if Aitken's δ^2 process is strictly followed for every component, i.e., the e_l transform is applied to every grid point, not only are the storage and arithmetic operations increased but the nonuniformity in λ_l and ϕ implicitly determined for different components may introduce additional noise to be filtered out. It proves to be less effective than the present procedure (cf. Sec. IV below). In passing, we note that the application of the δ^2 and the related ϵ algorithms of Wynn[26] have been crucial in Nieuwland's method of shock-free airfoil design.[31]

Among other acceleration techniques based on a power method with comparable simplicity is one in which the eigenvalue is shifted by changing the iterative matrix Q to $(1-p)^{-1}(Q-pI)$, where p is a constant.[4] This requires, however, *a priori* knowledge of the dominant eigenvalues.

IV. Accelerating the Line SOR Solution

As a test of the cyclic acceleration technique, the method has been applied to the line SOR solution to the Dirichlet problem for a unit square, using a standard central-difference formulation with a uniform mesh of $h=1/32$. Typical results, presented and discussed fully in Ref. 30, show that with an error norm of 10^{-3} to 10^{-4}, the iterations and computer times can be reduced by a factor of three to four through the acceleration technique based on the second-order transform. Application to the solution based on a nine-point central difference is found to be very successful. Examples illustrating the influence of the relaxation parameter, reverse sweep, variable mesh size (grid halving), and use of first- and second-order transforms with different estimates of λ_l are also studied in Ref. 30.

Unlike the Dirichlet problem, the partial differential equation governing a transonic flow is generally nonlinear and of the mixed type. In this case, the matrix Q governing the error vector of the relaxation solution is not known *a priori*; it is not possible to identify the optimum relaxation parameter for the efficient use of the SOR method. Therefore, application of cyclic acceleration techniques to this case should prove very useful. In this paper, we shall confine our discussion of results of application mainly to the solutions of the transonic small-disturbance theory.

A. Transonic Small-Disturbance Equation: Basic Line-Relaxation Program

The steady inviscid plane flow past a thin airfoil near sonic speed can be described by a perturbation velocity potential ϕ satisfying the Karman small-disturbance equation.[33] Let (x,y) be Cartesian coordinates with the x axis parallel to the freestream and c_* be a reference length taken below to be the half chord. The upper and lower airfoil surface is represented as $y=\tau\cdot c_*\cdot Y(x/c_*,\alpha/\tau,\pm 0)$, where τ is the thickness ratio and α the angle of attack. The Kármán equations and the boundary conditions can be written following Cole,[34] as

$$(K_c-(\gamma+1)\tilde{\phi}_{\bar{x}})\tilde{\phi}_{\bar{x}\bar{x}}+\tilde{\phi}_{\bar{y}\bar{y}}=0 \tag{26a}$$

$$\tilde{\phi}_{\bar{y}}(\bar{x},\pm 0)=Y'(\bar{x},\alpha/\tau,\pm 0) \quad \text{at} \quad |\bar{x}|\le 1 \tag{26b}$$

$$[\![\tilde{\phi}_{\bar{x}}]\!]=[\![\tilde{\phi}_{\bar{y}}]\!]=0 \text{ at } |\bar{x}|>1 \qquad \bar{y}=\pm 0 \tag{26c}$$

$$\tilde{\phi}_{\bar{x}},\ \tilde{\phi}_{\bar{y}}\to 0 \quad \text{as} \quad \bar{x}^2+\bar{y}^2\to\infty \tag{26d}$$

where $\bar{x}\equiv x/c_*$, $\bar{y}\equiv(M_\infty^2\tau)^{1/3}y/c_*$, $\tilde{\phi}\equiv(M_\infty/\tau)^{2/3}\phi/Uc_*$, ϕ is the perturbation potential, and

$$K_c\equiv(1-M_\infty^2)\cdot(M_\infty^2\tau)^{-2/3} \tag{27}$$

At the trailing edge which is assumed to be sharp, the Kutta-Joukowski condition is to be enforced. The pressure coefficient is evaluated as

$$c_p\equiv(p-p_\infty)/\tfrac{1}{2}\rho_\infty U^2=-2(\tau/M_\infty)^{2/3}\tilde{\phi}_{\bar{x}} \tag{28}$$

For a flow in an embedded supercritical region, the elliptic and hyperbolic regions are separated by the sonic boundary, $K_c-(\gamma+1)\tilde{\phi}_{\bar{x}}=0$, and by the shock $\bar{x}=\bar{x}^D(\bar{y})$, which satisfies the approximate jump conditions

$$\pm\langle -K_c+(\gamma+1)\tilde{\phi}_{\bar{x}}\rangle^{1/2}=[\![\tilde{\phi}_{\bar{y}}]\!]/[\![\tilde{\phi}_{\bar{x}}]\!]=-d\bar{x}^D/d\bar{y} \tag{29}$$

where $[\![\ ]\!]$ and $\langle\ \rangle$ respectively signify the jump and the average of quantities in question across a discontinuity.

In setting up the numerical procedure, the far-field condition (26d) is replaced by one over a rectangular boundary, say $\bar{x}=\pm 3$, $\bar{y}=\pm 6$. In cases of a high subsonic freestream (i.e., $K_c>0$) to be analyzed below, the value of $\tilde{\phi}$ on the far boundary can be described by the solution of the linearized form of Eq. (26a) for a vortex and doublet of unknown strengths, to be determined in the course of the iteration. The vortex strength is determined by the potential jump at the trailing edge; the doublet strength depends on the airfoil thickness distribution as well as the near-field nonlinear corrections, and has been estimated from data near the boundary with a least-square method at the end of each iteration.

The basic procedure, to which our acceleration and shock fitting methods is applied, follows that of Murman and Cole.[13] A central-difference operator with second-order accuracy is used in the elliptic region, and an implicit backward difference operator with a first-order accuracy is used in the hyperbolic region. In this basic program, a parabolic point operator corresponding to $\tilde{\phi}_{\bar{y}\bar{y}}=0$ is used at a grid point between the elliptic and the hyperbolic regions. The difference equations are solved by a line SOR method, assigning appropriate ω's to the two regions. The unknowns at points belonging to the same vertical line are solved simultaneously, while the line sweeps downstream. In solving the line SOR problem, the matrix is linearized by assigning values to the coefficient of P.D.E. (26a) from the most recent data; the resulting tridiagonal matrix for the line can readily be inverted.

The basic program used in the subsequent studies is written for an IBM 370/158. The grid has 81 point in $\bar{x}$ and 62 points in $\bar{y}$ (or 31 points in $\bar{y}$ if the problem has symmetry in $\bar{y}$), using unequal but gradually varying mesh size. The finest meshes $\Delta\bar{x}=0.05$ and $\Delta\bar{y}=0.04$ are assigned over $|\bar{x}|\le 1$ at $|\bar{y}|=0.02$.

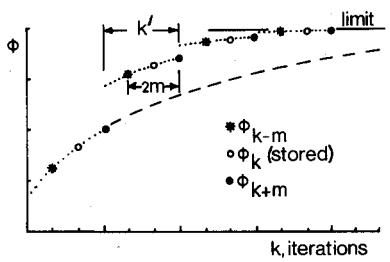


Fig. 1 Cyclic acceleration technique applied to an iterative solution, illustrated for the first-order transform. The cycle is repeated every k' iteration.

The truncation error in the difference equation system depends on the local mesh and is generally of the order $\Delta\tilde{x}$ and $(\Delta\tilde{y})^2$. There is, however, a re-expansion singularity at the shock root where the surface pressure is known to vary like x' $\log x'$, with $x' = \tilde{x} - \tilde{x}^D$.[35] This singularity causes a unit-order (100%) error in $\tilde{\phi}_{\tilde{x}\tilde{x}}$ near the shock root, but the relative errors in velocities belong to the order $\Delta\tilde{x} \log x'$. (Hence, there is no gain in using a second-order difference scheme near the shock, unless the re-expansion singularity is analytically accounted for.) With this in mind, our solution is locally no more accurate than

$$\mathcal{R} = O(\Delta_l\tilde{x} \log \tilde{x}', (\Delta_l\tilde{y})^2) \tag{30}$$

with $\Delta_l\tilde{x}$ and $\Delta_l\tilde{y}$ taken to be the smallest $\Delta\tilde{x}$ and $\Delta\tilde{y}$.

B. Examples: Parabolic-Arc Airfoil

As a first example for transonic flows, we study the acceleration of the line SOR solution for a parabolic-arc airfoil at zero incidence, for $K_c = 1.8$. The problem considered has a subsonic freestream but has an embedded supersonic region with an interior shock boundary. The same solution has been studied previously by Murman.[36]

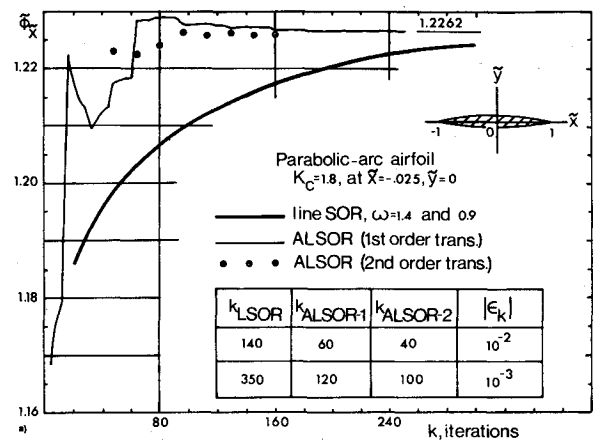
The unaccelerated solution is generated by the line SOR program with the smallest mesh being $\Delta\tilde{x} = 0.05$, $\Delta\tilde{y} = 0.04$, using over-relaxation in the subsonic region and under-relaxation in the supersonic region. Cyclic acceleration, using the first- as well as the second-order transforms, is applied. Each cycle consists of 16 iterations ($k' = 16$), with $m = 4$ (cf. Sec. IIID); the first cycle commences at $k = 9$ for the first-order procedure and $k = 17$ for the second-order procedure. The linear average quotient form $\overline{\lambda_l^m}$ is used for estimating λ_l^m, and the inner product form is used for estimating p_0 and p_1 (cf. Secs. IIIA and D; linear average forms for estimating p_j's have also been used without major differences.)

The convergence histories of solutions by the three different procedures are illustrated in Fig. 2a-2c for $K_c = 1.8$ at a representative point, for different combinations of the relaxation parameters ω in the elliptic and hyperbolic regions. In each case, the initial (trial) data are furnished by a sufficiently accurate solution to the same problem at $K_c = 2.1$, corresponding to a lower freestream Mach number. The unaccelerated line SOR solution is shown as a solid curve; the accelerated line SOR solution using the first-order transform (referred to as ALSOR-1 in the figures) is shown as a thin solid curve drawn through data from all iterations; for the solution accelerated by the second-order transform (referred to as ALSOR-2), only data points at the end of the cycles are shown (in open circles).

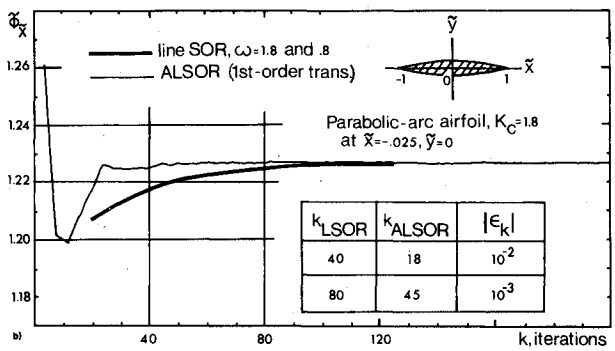
Figure 2a shows $\tilde{\phi}_{\tilde{x}}$ vs the number of iterations k and $\tilde{x} = 0.025$, $\tilde{y} = 0$, for the case in which $\omega = 1.4$ in the elliptic region and $\omega = 0.9$ in the hyperbolic region. The improvement in the convergence rate through cyclic application of the transforms is obvious. A reduction in iteration number by a factor of three to four is possible, depending on whether the accuracy requirement is set at 10^{-2} or 10^{-3} (cf. the table in Fig. 2a). We note that 10^{-2} is comparable to the truncation error of the difference equation; however, an error norm at least as small as 10^{-3} is required to indicate the convergence of the iterative solutions. To approach the limit with the 10^{-3} accuracy, the unaccelerated solution requires 350 or more iterations.‡‡

Figure 2b gives the results for a different pair of relaxation parameters; $\omega = 1.8$ in the elliptic region and $\omega = 0.8$ in the hyperbolic region. This combination turns out to give a much better convergence behavior. The line SOR solution (solid curve) approaches the limit within 10^{-2} in 40 iterations and within 10^{-3}. (We note that this set of calculations used a shorter cycle than those in Fig. 2a and c with $k' = 8$.) the convergence is so rapid in this case that use of the second-order transform is considered unnecessary.

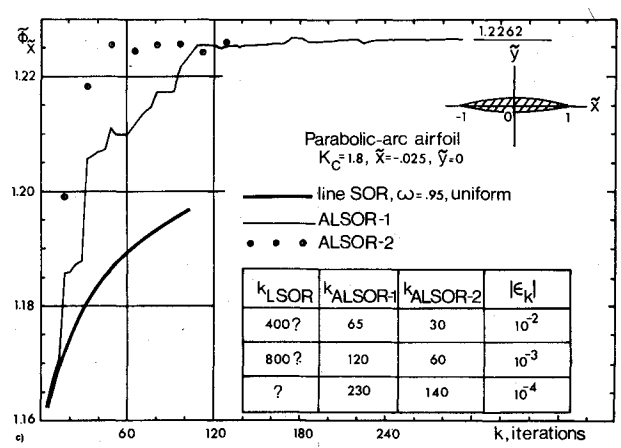
Most transonic flow computations to date have employed over-relaxation in the elliptic region and under-relaxation in the hyperbolic region. Figure 2c presents a case with a uniform relaxation parameter, $\omega = 0.95$ for the entire field. With this ω, convergence of the line SOR procedure becomes exceedingly slow—400 iterations or more would be needed to approach the limit within 10^{-2}. The power of the cyclic transform method to speed up convergence is most clearly



k_{LSOR}	$k_{ALSOR-1}$	$k_{ALSOR-2}$	$\lvert\epsilon_k\rvert$
140	60	40	10^{-2}
350	120	100	10^{-3}



k_{LSOR}	k_{ALSOR}	$\lvert\epsilon_k\rvert$
40	18	10^{-2}
80	45	10^{-3}



k_{LSOR}	$k_{ALSOR-1}$	$k_{ALSOR-2}$	$\lvert\epsilon_k\rvert$
400?	65	30	10^{-2}
800?	120	60	10^{-3}
?	230	140	10^{-4}

Fig. 2 Test of cyclic acceleration technique on line SOR solution to a supercritical transonic flow over a parabolic-arc airfoil at zero incidence with $K_c = 1.8$ at $\tilde{x} = 0.025$ and $\tilde{y} = 0$, using first- and second-order transforms: a) $\omega = 1.4$ and 0.9; b) $\omega = 1.8$ and 0.8; c) $\omega = 0.95$, uniform.

‡‡If error norm comparable to 10^{-4} or 10^{-5} is deemed necessary (to reject spurious solution), the gain by the present method will appear even greater, since $k \propto N \lvert\log\epsilon_k\rvert$.

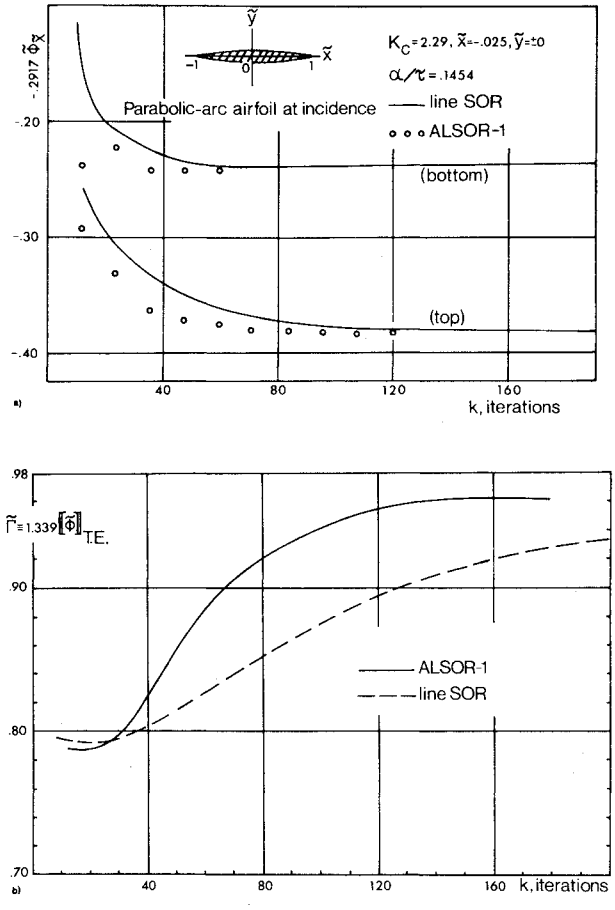


Fig. 3 Test of cyclic acceleration technique on line SOR solution with $\omega = 1.8$ and 0.8 for a supercritical transonic flow over a parabolic-arc airfoil at incidence with $K_c = 2.29$, $\alpha/\tau = 0.1454$, using first-order transform: a) convergence history of surface speed near midchord $\bar{x} = -0.025$, $\bar{y} = \pm 0$, b) convergence history of the circulation.

demonstrated in this case. At three levels of accuracy (10^{-2}, 10^{-3}, 10^{-4}), the accelerated solution using the first-order transform approaches the limit in 65, 120, and 230 iterations, respectively. Application of the second-order transform reduces the iterations further to 30, 60, and 140 (cf. table in Fig. 2c).

C. Example with Circulation

Satisfactory convergence of the iterative solutions to the 2-D transonic problem involving lift requires 250-1200 iterations according to published works.[14,18] We shall examine below the convergence characteristics of a line SOR solution and its accelerated version for a parabolic-arc airfoil at incidence, with $K_c = 2.29$ and $\alpha/\tau = 0.1454$, corresponding to $M_\infty = 0.848$, a 6% thickness ratio, and a 0.5° angle of attack. The line SOR procedure uses $\omega = 1.8$ and 0.8 for the subsonic and supersonic regions, respectively. The cyclic acceleration procedure employs the first-order transform with $m = 2$, $k' = 12$ iterations/cycle, and λ_l^m being estimated by the inner product from Eq. (25). Typical convergence histories for the velocity perturbation on the top and bottom airfoil surfaces are shown in Fig. 3a for a point near the midchord, $\bar{x} = -0.025$ and $\bar{y} = \pm 0$. The unaccelerated solution is given as a solid curve and the accelerated solution in open circles. For the latter, only data points at the conclusion of each cycle are shown. The improvement in convergence rate through the cyclic method is quite evident, although not exceedingly great for this particular point. The usefulness of the method is more clearly shown in Fig. 3b, where the convergence history for the circulation $\tilde{\Gamma} = [\![\tilde{\phi}]\!]_{T.E.}$ is presented. The accelerated solution approaches the limit within 10^{-3} after 150 iterations, whereas for the *same* accuracy of the line SOR without acceleration takes 800 iterations.

In passing, we observe that the shock jumps deduced from the converged solutions in all preceeding examples, as in most solutions based on the original SOR programs of Murman and Cole,[13] do not fully satisfy the jump relation Eq. (29) (cf. Discussion in Ref. 20). In the last example considered, the shock strength is found to be 20% too low, and use of finer grids does not prove to be helpful. These line SOR programs may nevertheless be improved with the use of a shock-point operator,[36] or with shock-fitting,[30] to be studied in a sequel paper.[37] Our technique proves to be effective in accelerating both versions of the improved SOR solutions.

V. Concluding Remarks

In this paper, we have studied techniques of accelerating the relaxation methods. Essential in the acceleration technique is a transformation applied cyclically to the iterative solutions, which generates a new set of data closer to the convergence limit for iterations in the next cycle. The key formula has much in common with the "e_m" transform of Shanks[10] and Aitken's δ^2 process[11,12] but derives its theoretical basis from the power method.[29]

Cyclic techniques based on the first- and second-order transforms have been tested in a model Dirichlet problem and the transonic airfoil problem. The results have demonstrated the effectiveness of the technique in speeding up the convergence of the line relaxation solutions for the elliptic as well as quasilinear, mixed-type problems, with different choices of the relaxation parameters and sweep directions. In most cases studied, reduction in the total iteration number and computation time by a factor of two to four can be achieved, depending on the accuracy requirement and other considerations. The additional calculations in the application of the acceleration technique amounts to no more than 5% of the total work.

Lyusternick[32] and Wilkinson[4] have derived formulas identifiable with those of the first- and second-order transforms, but their analyses cease to be valid as the moduli of the dominant eigenvalues of the iterative matrix approach unity. The present study has contributed a more critical error study for the transforms, which allows for the moduli of a set of closely-spaced eigenvalues to approach unity. Implicit in the transform is the assumption of the existence of dominant eigenvalues whose moduli are larger than those of the rest. Situations do arise wherein this stipulation is not met (e.g., if $\omega > \omega_{opt}$ in the model Dirichlet problem). In these circumstances the remedy is to readjust the relaxation parameter or to introduce reverse sweeps in the basic line SOR program.

Applications of the cyclic method have been limited to accelerating the line SOR solutions of Murman and Cole for a parabolic-arc airfoil at small incidence. Similar improvement in convergence properties may be expected for more complicated 2-D and 3-D relaxation programs for elliptic and mixed-type problems, with and without shock fitting, including the shock point-operator methods[36] and its extension. Most recently, the acceleration method discussed here has been applied with considerable success to the calculation of transonic nacelle flow by Caughey and Jameson.[39]§§ The present approach possesses also the potentiality for speeding up certain pseudo-unsteady finite-difference methods.[20]

Owing to the storage limitation, only the most rudimentary among the transforms, e_1 and e_2, have been used. Recursive application of Wynn's ϵ algorithm without this storage problem is yet to be explored.[26]

§§The more spectacular results demonstrated in Ref. 39 may be attributed partly to the exceedingly small error norm $|\epsilon_k|$ considered, which ranges from 10^{-4} to 10^{-9}, noting that $k \propto N |\log \epsilon_k|$.

Acknowledgments

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